Pauli Matrices

The Pauli matrices are three 2×2 matrices conventionally defined as follows.

$$\sigma_1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli matrices have the following useful properties.

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

These properties can be combined to derive the following result.

$$\begin{aligned} \sigma_i\sigma_j &= \frac{1}{2}(\sigma_i\sigma_j - \sigma_j\sigma_i) + \frac{1}{2}(\sigma_i\sigma_j + \sigma_j\sigma_i) \\ &= \frac{1}{2}[\sigma_i, \sigma_j] + \frac{1}{2}\{\sigma_i, \sigma_j\} \\ &= \delta_{ij} + i\epsilon_{ijk}\sigma_k \end{aligned}$$

Other properties that one can determine by examining the above definitions are as follows.

$$\sigma_i^\dagger = \sigma_i \qquad \mathrm{tr}(\sigma_i) = 0 \qquad \det(\sigma_i) = -1$$

It is also useful to note the symmetry of each of the Pauli matrices.

$$\sigma_1^T = \sigma_1 \qquad \sigma_2^T = -\sigma_2 \qquad \sigma_3^T = \sigma_3$$

We can write this compactly as follows.

$$\sigma_i^T = (-1)^{i+1}\sigma_i$$