

SU(2) Traces

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The Pauli matrices satisfy the following.

$$\text{tr}(\sigma^i) = 0 \quad \sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk} \sigma^k$$

These properties can be used to simplify traces of products of Pauli matrices. The first few traces are as follows.

$$\begin{aligned}\text{tr}(\sigma^i) &= 0 \\ \text{tr}(\sigma^i \sigma^j) &= 2\delta^{ij} \\ \text{tr}(\sigma^i \sigma^j \sigma^k) &= 2i\epsilon^{ijk} \\ \text{tr}(\sigma^i \sigma^j \sigma^k \sigma^l) &= 2(\delta^{ij} \delta^{kl} - \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk})\end{aligned}$$

These results can be derived as follows.

$$\begin{aligned}\text{tr}(\sigma^i \sigma^j) &= \text{tr}(\delta^{ij} + i\epsilon^{ija} \sigma^a) = 2\delta^{ij} \\ \text{tr}(\sigma^i \sigma^j \sigma^k) &= \text{tr}((\delta^{ij} + i\epsilon^{ija} \sigma^a) \sigma^k) = i\epsilon^{ija} \text{tr}(\sigma^a \sigma^k) = i\epsilon^{ija} 2\delta^{ak} = 2i\epsilon^{ijk} \\ \text{tr}(\sigma^i \sigma^j \sigma^k \sigma^l) &= \text{tr}((\delta^{ij} + i\epsilon^{ija} \sigma^a) \sigma^k \sigma^l) = \delta^{ij} \text{tr}(\sigma^k \sigma^l) + i\epsilon^{ija} \text{tr}(\sigma^a \sigma^k \sigma^l) = \\ &= \delta^{ij} 2\delta^{kl} + i\epsilon^{ija} 2i\epsilon^{akl} = 2\delta^{ij} \delta^{kl} - 2\epsilon^{ija} \epsilon^{kla} = 2\delta^{ij} \delta^{kl} - 2(\delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk}) = \\ &= 2(\delta^{ij} \delta^{kl} - \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk})\end{aligned}$$